PHYC 511
Spring 2018

Lecture 26 04/30/2018

Radiation (Cont'd)

Radiation for Relativistically Moving Charges

Let a charge q move on the trajectory $\vec{x} = \vec{r}(t)$. Its instantaneous velocity then is $\vec{v}(t) = \frac{d\vec{r}}{dt}$, and the corresponding charge density and current density are given by:

$$\rho(\vec{x},t) = q\,\delta^{(3)}(\vec{x} - \vec{r}(t)) \quad , \quad \vec{J}(\vec{x},t) = q\,\vec{v}(t)\,\delta^{(3)}(\vec{x} - \vec{r}(t))$$

As we saw, in the Lorenz gauge, the radiated potentials are:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0} \iint \frac{\rho(\vec{x}\,',t')\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}\,'|}{c}\right)}{|\vec{x} - \vec{x}\,'|}\, d^3x'\, dt'$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi} \iint \frac{\vec{J}(\vec{x}\,',t')\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}\,'|}{c}\right)}{|\vec{x} - \vec{x}\,'|}\, d^3x'\, dt'$$

Here, we have used the retarded Green's function.

We consider deriving Φ in detail, and the situation for $\vec{A}$ will be similar. We have:

$$\Phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{\delta\left(t'-t+\frac{|\vec{x}-\vec{r}(t')|}{c}\right)}{|\vec{x}-\vec{r}(t')|}\,dt'$$

Recall that in general:

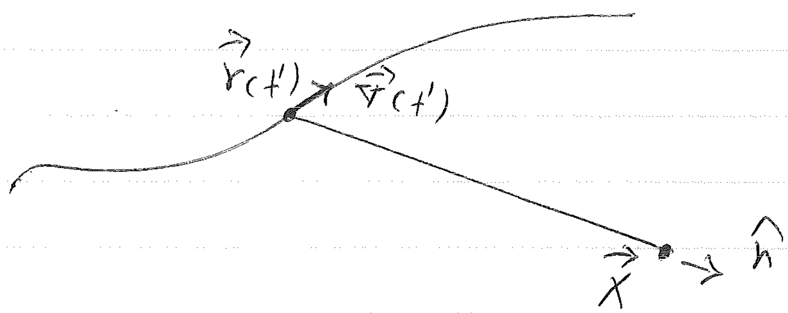
$$\delta(f(t')) = \frac{\delta(t'-t_0)}{|f'(t_0)|} \qquad (f(t_0)=0)$$

In this case:

$$f(t') = t'-t+\frac{|\vec{x}-\vec{r}(t')|}{c} = 0 \Rightarrow t' = t-\frac{|\vec{x}-\vec{r}(t')|}{c}$$

$$\frac{d}{dt'}\left(t'-t+\frac{|\vec{x}-\vec{r}(t')|}{c}\right) = 1+\frac{1}{c}\frac{d}{dt'}\sqrt{\vec{x}\cdot\vec{x}+\vec{r}\cdot\vec{r}-2\vec{x}\cdot\vec{r}} = 1-$$

$$\frac{\vec{v}(t')\cdot(\vec{x}-\vec{r}(t'))}{c|\vec{x}-\vec{x}'|} = 1-\vec{\beta}\cdot\hat{n}$$



Where:

$$\vec{\beta} \equiv \frac{\vec{v}(t')}{c}, \qquad \hat{n} \equiv \frac{\vec{x}-\vec{r}}{|\vec{x}-\vec{r}|}$$

Therefore ($R \equiv |\vec{x}-\vec{r}|$):

$$\boxed{\Phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0}\left.\frac{1}{R(1-\vec{\beta}\cdot\hat{n})}\right|_{ret}}$$

Similarly:

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0 q c}{4\pi}\left.\frac{\vec{\beta}}{R(1-\vec{\beta}\cdot\hat{n})}\right|_{ret}}$$

These are called the <u>Lienard-Wiechert potentials.</u>

There is a nice physical explanation for the $1-\vec{\beta}\cdot\hat{n}$ factor in the denominator based on relativistic kinematics. Lets take the point charge as a rod of length L with uniform linear charge density $\frac{q}{L}$ in the limit that $L \to 0$. Consider the case when the rod is moving toward the observer at velocity $\vec{v}$:

In order for the fields from the back (B) and front (F) ends to reach the observer at the same time, they must be emitted at different moments of time t'_B and t'_F, where:

$$\Delta t' = t'_F - t'_B = \frac{L}{c-v}$$

This gives the rod an apparently larger length, and hence a net effective charge $\frac{q}{1-\beta}$. The opposite happens when the charge is moving away from the observer, resulting in an effective charge $\frac{q}{1+\beta}$. For a general motion, we get a factor of $1-\vec{\beta}\cdot\hat{n}$ in the denominator.

Next, let us find the electric and magnetic fields in the radiation zone. The exact results are complicated, but in the radiation zone we can keep the leading order terms $\propto \frac{1}{R}$. In this limit, all derivatives of the $\frac{1}{R}$ factor in Φ and $\vec{A}$ can be neglected as they result in terms $\propto \frac{1}{R^2}$.

Starting with the electric field, we have:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Then:

$$-\vec{\nabla}\Phi \simeq \frac{-q}{4\pi\epsilon_0 R_{ret}} \vec{\nabla}\left(\frac{1}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret} \quad , \quad -\frac{\partial \vec{A}}{\partial t} \simeq \frac{-\mu_0 q c}{4\pi R_{ret}} \frac{\partial}{\partial t}\left(\frac{\vec{\beta}}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret}$$

$$\vec{\nabla}\left(\frac{1}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret} = \vec{\nabla}t' \frac{\partial}{\partial t'}\left(\frac{1}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret}$$

$$\vec{\nabla}t' = \vec{\nabla}\left(t - \frac{|\vec{x}-\vec{r}(t')|}{c}\right) = -\frac{1}{c}\vec{\nabla}\left(\sqrt{\vec{x}\cdot\vec{x} + \vec{r}\cdot\vec{r} - 2\vec{x}\cdot\vec{r}}\right) =$$

$$-\frac{1}{c}\frac{\vec{x}-\vec{r}(t')}{|\vec{x}-\vec{r}(t')|}+\frac{\vec{x}-\vec{r}(t')\cdot\dot{\vec{r}}}{c|\vec{x}-\vec{r}(t')|}\vec{\nabla}t' = -\frac{1}{c}\hat{n}+(\hat{n}\cdot\vec{\beta})\vec{\nabla}t' \Rightarrow$$

$$\vec{\nabla}t' = \left.\frac{-\frac{\hat{n}}{c}}{1-\hat{n}\cdot\vec{\beta}}\right|_{ret}$$

Also:

$$\frac{\partial}{\partial t'}\left(\frac{1}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret} = \frac{1}{1-\vec{\beta}\cdot\hat{n}}\left(\hat{n}\cdot\dot{\vec{\beta}}+\dot{\hat{n}}\cdot\vec{\beta}\right) \underset{\substack{\downarrow \\ \text{neglecting} \\ O(\frac{1}{R^2})\text{ terms}}}{\simeq} \frac{-\dot{\vec{r}}}{R} \Rightarrow$$

$$\frac{\partial}{\partial t'}\left(\frac{1}{\vec{\beta}\cdot\hat{n}}\right)_{ret} = \left.\frac{\hat{n}\cdot\dot{\vec{\beta}}}{(1-\vec{\beta}\cdot\hat{n})^2}\right|_{ret}$$

Similarly:

$$\frac{\partial}{\partial t}\left(\frac{\vec{\beta}}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left(\frac{\vec{\beta}}{1-\vec{\beta}\cdot\hat{n}}\right)_{ret}$$

$$\frac{\partial t'}{\partial t} = \frac{\partial(t-\frac{R}{c})}{\partial t} = 1-\frac{1}{c}\frac{\partial}{\partial t}\sqrt{\vec{x}\cdot\vec{x}+\vec{r}\cdot\vec{r}-2\vec{x}\,\vec{r}} = 1+\frac{1}{c}\frac{(\vec{x}-\vec{r})\cdot\dot{\vec{r}}}{R}\frac{\partial t'}{\partial t}$$

$$\Rightarrow \frac{\partial t'}{\partial t} = \frac{1}{1-\vec{\beta}\cdot\hat{n}}$$

And:

$$\frac{\partial}{\partial t'}\left(\frac{\vec{\beta}}{1-\vec{\beta}\cdot\hat{n}}\right) = \frac{\dot{\vec{\beta}}}{1-\vec{\beta}\cdot\hat{n}}+\frac{\vec{\beta}\,(\dot{\vec{\beta}}\cdot\hat{n})}{(1-\vec{\beta}\cdot\hat{n})^2}$$

Hence, putting every thing together, we find:

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 c R_{ret}} \frac{\left[\dot{\vec{\beta}}\cdot\hat{n} - (1-\vec{\beta}\cdot\hat{n})\dot{\vec{\beta}} - (\dot{\vec{\beta}}\cdot\hat{n})\vec{\beta}\right]}{(1-\vec{\beta}\cdot\hat{n})^3}}$$

We note that $\vec{E}$ vanishes when $\dot{\vec{\beta}}=0$. This is in agreement with expectation as only accelerating charges can radiate in vacuum.